RETRODICTIVELY OPTIMAL LOCALISATIONS IN PHASE SPACE

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Abstract

In a previous paper it was shown that the distribution of measured values for a retrodictively optimal simultaneous measurement of position and momentum is always given by the initial state Husimi function. This result is now generalised to retrodictively optimal simultaneous measurements of an arbitrary pair of rotated quadratures $\hat{x}_{\theta_1}$ and $\hat{p}_{\theta_2}$. It is shown, that given any such measurement, it is possible to find another such measurement, informationally equivalent to the first, for which the axes defined by the two quadratures are perpendicular. It is further shown that the distribution of measured values for such a meaurement belongs to the class of generalised Husimi functions most recently discussed by Wünsche and Bužek. The class consists of the subset of Wódkiewicz's operational probability distributions for which the filter reference state is a squeezed vaccuum state.

1. Introduction

During the last few years there has been considerable progress in the problem of simultaneously measuring both the position and the momentum of a quantum mechanical system[1, 2]. In several recent publications [3, 4, 5] we have discussed how to characterise the accuracy of, and disturbance caused by such measurements. One approach to the problem is that based on the concept of an "unsharp observable" [6, 7, 8, 9]. This approach has recently been criticised by Uffink [10]. In the papers just mentioned we took a rather different approach, based on methods developed by Braginsky and Khalili [11]. It appears to us that these methods have certain advantages, both conceptually (they clarify what is meant by the term "accuracy" in a quantum mechanical context [3]), and practically (they facilitate the calculations [4]).

Another advantage of these methods is that they give additional insight into the physical significance of the Husimi function [12, 13]. The fact that the Husimi function describes the distribution of measured values for many *particular cases* of joint measurement processes is, of course, well known [1, 2]. In ref. [5] (also see Prugovečki and Ali [8]) we showed that the Husimi function actually has a much stronger, *universal* property: namely, it gives the distribution of results for *any* retrodictively optimal measurement process (*i.e.* any process which is retrodictively unbiased, and which maximises the retrodictive accuracy).

The fact that the Husimi function gives the distribution of results whenever the measurement is retrodictively optimal, and is otherwise independent of the details of the particular process employed, could be interpreted to mean that the Husimi function plays the same role for joint measurements of x and p that is played by the function $|\langle x \mid \psi\rangle|^2$ for single measurements of x only.

The purpose of this paper is to show that a similar universal property holds for generalised Husimi functions of the form

$$Q_{\text{gen}}(x,p) = \frac{1}{\pi}\int dx'dp' \, \exp\left[-\left(a(x-x')^2 + 2c(x-x')(p-p') + b(p-p')^2\right)\right] \times W(x',p') \tag{1}$$

where a, b and c are real, $a, b \geq 0$, $ab - c^2 = 1$, and where W is the Wigner function. These functions are operational distributions of the type defined by Wódkiewicz [6, 9, 14, 15]. They are the distributions which result when the filter reference state (or "quantum ruler") is an arbitrary squeezed vacuum state [16]. They have been discussed by Halliwell [17], Wünsche [18] and Wünsche and Bužek [19].

In ref. [5] we considered retrodictively optimal measurements of x and p. However, this is clearly not the only way to determine the location of a system in phase space. What would happen if, instead of determining x and p, one were to make a retrodictively optimal measurement of an arbitrary pair of rotated quadratures, not necessarily at 90^{o} (see Fig. 1)? Such measurements are possible (using a suitably modified form of homodyne detection [2], for example). We will show that the outcome of such a measurement is always described by a generalised Husimi function of the kind defined by Eq. (1). As with our previous result this is a universal statement: it only depends on the measurement being retrodictively optimal, and is otherwise independent of the details of the particular process employed.

The main difficulty in proving this result comes from the fact that the measurements we consider are characterised by three independent parameters [namely,two angles θ, ϕ to specify the oblique coordinate system (see Fig. 1), and a parameter λ to specify the relative accuracy of the measurements of the two quadratures]. On the other hand it only needs two parameters to specify a distribution of the type defined by Eq. (1). It follows, that corresponding to any given distribution,

there is an infinite set of informationally equivalent measurements. It is the problem of characterising these sets, and giving a precise definition of "informational equivalence," which will mainly concern us in the following.

One might also ask what is the significance of distributions which are like the ones considered in this paper in that they are obtained from the Wigner function by smoothing it with a Gaussian convolution, but in which the determinant $ab-c^2 > 1$ (corresponding to an impure filter reference state) [15, 17, 18, 19, 20, 21]. In ref. [4] we showed that, in the special case of the Arthurs-Kelly process, such functions describe the outcome of measurements producing the smallest possible amount of disturbance for a given, sub-optimal degree of accuracy. It is natural to wonder whether this property is also universal, and whether it also generalises to the case of simultaneous measurements of an arbitrary pair of rotated quadratures. However, that is a question which we leave to the future.

2. Linear Canonical Transformations

Squeezed states arise as a result of making linear canonical transformations of the creation and annihilation operators [16]. We begin by describing the parameterisation of these transformations which will be employed in the sequel.

Consider a system, having one degree of freedom, with position $\hat{x}$ and momentum $\hat{p}$. In some applications $\hat{x}$, $\hat{p}$ are dimensionless to begin with. If not they can be made dimensionless, by making the replacements $\hat{x} \to \frac{1}{l}\hat{x}$, $\hat{p} \to \frac{l}{\hbar}\hat{p}$, where l is a fixed, in general arbitrary constant having the dimensions of length. In the sequel we will always assume that this has been done, so that $[\hat{x}, \hat{p}] = i$.

We are interested in transformations of the form

$$\begin{pmatrix} \hat{x}_M \\ \hat{p}_M \end{pmatrix} = M \begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix}$$

where [16] M belongs to the group $\mathrm{SL}(2,\mathbb{R})$ [which is isomorphic to $\mathrm{Sp}(2,\mathbb{R})$]. The group may be parameterised as follows. Given any matrix $M \in \mathrm{SL}(2,\mathbb{R})$ there exist unique θ in the range $-\pi < \theta \leq \pi$ and unique ϕ in the range $-\frac{\pi}{4} < \phi < \frac{\pi}{4}$ such that

$$M = \begin{pmatrix} \alpha \cos(\theta+\phi) & \alpha \sin(\theta+\phi) \\ -\beta \sin(\theta-\phi) & \beta \cos(\theta-\phi) \end{pmatrix}$$

for suitable positive constants α, β (see Fig. 1). The requirement that $\det M = 1$ means that $\alpha\beta \cos 2\phi = 1$. We may therefore write

$$M = \sqrt{\sec 2\phi} \begin{pmatrix} \frac{1}{\lambda} \cos(\theta+\phi) & \frac{1}{\lambda} \sin(\theta+\phi) \\ -\lambda \sin(\theta-\phi) & \lambda \cos(\theta-\phi) \end{pmatrix} \tag{2}$$

for unique λ in the range $0 < \lambda < \infty$. We will refer to θ as the rotation, ϕ as the obliquity and λ as the resolution.

The matrix M defines a metric on phase space, with metric tensor $M^{\mathrm{T}}M$:

$$ds_M^2 = dx_M^2 + dp_M^2 = \begin{pmatrix} dx & dp \end{pmatrix} M^{\mathrm{T}} M \begin{pmatrix} dx \\ dp \end{pmatrix} \tag{3}$$

(M^{T} being the transpose of M). For a given system some choices of the matrix M, and therefore some choices of metric, will be more natural than others. However, if one wants to keep the discussion completely general, so that the detailed nature of the system is left unspecified, then one must regard the different choices for M as all being on the same footing—corresponding to the well-known fact, that there is (in general) no natural metric on phase space.

From this point of view, the choice of one particular conjugate pair $\hat{x}$ and $\hat{p}$ as basic, $\hat{x}_M$ and $\hat{p}_M$ being defined in terms of them, must be regarded as arbitrary. Assignments of angle also depend on the choice of metric tensor, and must likewise

be regarded as arbitrary. It follows, that in a general context (though perhaps not in applications to a particular type of system), no fundamental significance attaches to the distinction between oblique axes ($\phi \neq 0$) and perpendicular axes ($\phi = 0$).

It will be shown below that to each $M \in \mathrm{SL}(2,\mathbb{R})$ there corresponds a retrodictively optimal measurement; and that two such matrices define the same metric if and only if the corresponding measurements are informationally equivalent.

M has the decomposition

$$M = T_\lambda \, S_\phi \, R_\theta \tag{4}$$

where

$$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad S_\phi = \sqrt{\sec 2\phi} \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \qquad T_\lambda = \begin{pmatrix} \frac{1}{\lambda} & 0 \\ 0 & \lambda \end{pmatrix} \tag{5}$$

Define $\hat{a} = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p})$. Let $\hat{U}_\theta$ be the unitary rotation operator, and $\hat{V}_\phi$, $\hat{W}_\lambda$ the unitary squeeze operators [16] defined by

$$\begin{aligned} \hat{U}_\theta &= \exp\left[-i\theta \, \hat{a}^\dagger \hat{a} \right] \\ \hat{V}_\phi &= \exp\left[\tfrac{i}{2} \tanh^{-1}(\tan\phi) \left(\hat{a}^2 + \hat{a}^{\dagger\,2} \right) \right] \\ \hat{W}_\lambda &= \exp\left[\tfrac{1}{2} \ln\lambda \left(\hat{a}^2 - \hat{a}^{\dagger\,2} \right) \right] \end{aligned}$$

Then

$$\hat{U}_\theta^\dagger \begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix} \hat{U}_\theta = R_\theta \begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix} \tag{6}$$

$$\hat{V}_\phi^\dagger \begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix} \hat{V}_\phi = S_\phi \begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix} \tag{7}$$

$$\hat{W}_\lambda^\dagger \begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix} \hat{W}_\lambda = T_\lambda \begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix} \tag{8}$$

Hence

$$\begin{pmatrix} \hat{x}_M \\ \hat{p}_M \end{pmatrix} = \hat{U}_\theta^\dagger \hat{V}_\phi^\dagger \hat{W}_\lambda^\dagger \begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix} \hat{W}_\lambda \hat{V}_\phi \hat{U}_\theta$$

3. Informationally Equivalent Measurements

The purpose of this section is to show how the set of retrodictively optimal measurements divides into subsets of informationally equivalent measurements.

Suppose that we make a retrodictively optimal measurement of the conjugate observables $\hat{x}_M$, $\hat{p}_M$, of the kind described in ref. [5]. Let $\hat{\mu}_{\mathrm{X}M}$ and $\hat{\mu}_{\mathrm{P}M}$ be the pointer observables giving the results of the measurements of $\hat{x}_M$ and $\hat{p}_M$ respectively. Let $\hat{U}_{\mathrm{meas}}$ be the unitary evolution operator describing the measurement interaction, and let $\hat{\epsilon}_{\mathrm{X}M\mathrm{i}}$, $\hat{\epsilon}_{\mathrm{P}M\mathrm{i}}$ be the retrodictive error operators

$$\begin{aligned} \hat{\epsilon}_{\mathrm{X}M\mathrm{i}} &= \hat{U}_{\mathrm{meas}}^\dagger \, \hat{\mu}_{\mathrm{X}M} \hat{U}_{\mathrm{meas}} - \hat{x}_M \\ \hat{\epsilon}_{\mathrm{P}M\mathrm{i}} &= \hat{U}_{\mathrm{meas}}^\dagger \, \hat{\mu}_{\mathrm{P}M} \hat{U}_{\mathrm{meas}} - \hat{p}_M \end{aligned} \tag{9}$$

as defined in ref. [5]. If the measurement is retrodictively optimal there exists [5] fixed τ such that

$$\begin{aligned} \left(\left\langle \psi \otimes \phi_{\mathrm{ap}} \right| \hat{\epsilon}_{\mathrm{X}M\mathrm{i}}^2 \left| \psi \otimes \phi_{\mathrm{ap}} \right\rangle \right)^{\frac{1}{2}} &= \Delta_{\mathrm{ei}} x_M = \frac{\tau}{\sqrt{2}} \\ \left(\left\langle \psi \otimes \phi_{\mathrm{ap}} \right| \hat{\epsilon}_{\mathrm{P}M\mathrm{i}}^2 \left| \psi \otimes \phi_{\mathrm{ap}} \right\rangle \right)^{\frac{1}{2}} &= \Delta_{\mathrm{ei}} p_M = \frac{1}{\sqrt{2}\,\tau} \end{aligned}$$

for every normalised initial system state $|\psi\rangle$ (where $|\phi_{\mathrm{ap}}\rangle$ is the initial apparatus state, and $\Delta_{\mathrm{ei}} x_M$, $\Delta_{\mathrm{ei}} p_M$ are the maximal rms errors of retrodiction [3, 5]).

There is no loss of generality in confining ourselves to balanced measurements, for which $\tau = 1$. In fact, suppose that $\tau \neq 1$. Define

$$N = \begin{pmatrix} \frac{1}{\tau} & 0 \\ 0 & \tau \end{pmatrix} M$$

It can be seen that measuring the observables $\hat{x}_M$, $\hat{p}_M$ to retrodictive accuracies $\pm\frac{\tau}{\sqrt{2}}$ and $\pm\frac{1}{\sqrt{2}\,\tau}$ respectively is equivalent to measuring the observables $\hat{x}_N$, $\hat{p}_N$ both to the same retrodictive accuracy $\pm\frac{1}{\sqrt{2}}$.

Let M be the matrix with rotation θ, obliquity ϕ and resolution λ, as in Eq. (2). Define

$$\hat{x}_{\theta+\phi} = \cos(\theta+\phi)\hat{x} + \sin(\theta+\phi)\hat{p} \qquad\qquad \hat{p}_{\theta-\phi} = -\sin(\theta-\phi)\hat{x} + \cos(\theta-\phi)\hat{p}$$

Then

$$\hat{x}_M = \frac{\sqrt{\sec 2\phi}}{\lambda}\,\hat{x}_{\theta+\phi} \qquad\qquad \hat{p}_M = \lambda\sqrt{\sec 2\phi}\,\hat{p}_{\theta-\phi}$$

It follows, that making a retrodictively optimal, balanced measurement of $\hat{x}_M$, $\hat{p}_M$ is equivalent to making a retrodictively optimal measurement of $\hat{x}_{\theta+\phi}$, $\hat{p}_{\theta-\phi}$ to accuracies $\pm\frac{\lambda}{\sqrt{2\sec 2\phi}}$ and $\pm\frac{1}{\lambda\sqrt{2\sec 2\phi}}$ respectively. This equivalence means that there is associated, to each retrodictively optimal measurement of a pair $\hat{x}_{\theta+\phi}$, $\hat{p}_{\theta-\phi}$, a unique matrix $\in \mathrm{SL}(2,\mathbb{R})$.

Let M, M' be any two matrices $\in \mathrm{SL}(2,\mathbb{R})$, with parameter values θ, ϕ, λ and θ', ϕ', λ' respectively. Suppose that we make a retrodictively optimal, balanced measurement of the observables $\hat{x}_M$, $\hat{p}_M$. Let $\hat{\mu}_{\mathrm{X}M}$, $\hat{\mu}_{\mathrm{P}M}$ be the pointer observables representing the result of this measurement. Define new pointer observables $\hat{\mu}_{\mathrm{X}M'}$, $\hat{\mu}_{\mathrm{P}M'}$:

$$\begin{pmatrix} \hat{\mu}_{\mathrm{X}M'} \\ \hat{\mu}_{\mathrm{P}M'} \end{pmatrix} = L \begin{pmatrix} \hat{\mu}_{\mathrm{X}M} \\ \hat{\mu}_{\mathrm{P}M} \end{pmatrix} \tag{10}$$

where L is the matrix

$$L = \begin{pmatrix} L_{xx} & L_{xp} \\ L_{px} & L_{pp} \end{pmatrix} = M' M^{-1}$$

$\hat{\mu}_{\mathrm{X}M'}$, $\hat{\mu}_{\mathrm{P}M'}$ provide a measurement of $\hat{x}_{M'}$, $\hat{p}_{M'}$. We now ask: what is the condition for this measurement to be retrodictively optimal and balanced, the same as the measurement of $\hat{x}_M$, $\hat{p}_M$?

The retrodictive error operators for the M'-measurement are given by

$$\begin{pmatrix} \hat{\epsilon}_{\mathrm{X}M'\mathrm{i}} \\ \hat{\epsilon}_{\mathrm{P}M'\mathrm{i}} \end{pmatrix} = \begin{pmatrix} \hat{U}^{\dagger}_{\mathrm{meas}}\,\hat{\mu}_{\mathrm{X}M'}\hat{U}_{\mathrm{meas}} - \hat{x}_{M'} \\ \hat{U}^{\dagger}_{\mathrm{meas}}\,\hat{\mu}_{\mathrm{P}M'}\hat{U}_{\mathrm{meas}} - \hat{p}_{M'} \end{pmatrix} = L \begin{pmatrix} \hat{\epsilon}_{\mathrm{X}M\mathrm{i}} \\ \hat{\epsilon}_{\mathrm{P}M\mathrm{i}} \end{pmatrix}$$

[see Eq. (9)]. Therefore

$$\begin{aligned} \langle \hat{\epsilon}^2_{\mathrm{X}M'\mathrm{i}} \rangle &= L_{xx}^2 \langle \hat{\epsilon}^2_{\mathrm{X}M\mathrm{i}} \rangle + L_{xp}^2 \langle \hat{\epsilon}^2_{\mathrm{P}M\mathrm{i}} \rangle + L_{xx}L_{xp} \langle (\hat{\epsilon}_{\mathrm{X}M\mathrm{i}}\hat{\epsilon}_{\mathrm{P}M\mathrm{i}} + \hat{\epsilon}_{\mathrm{P}M\mathrm{i}}\hat{\epsilon}_{\mathrm{X}M\mathrm{i}}) \rangle \\ \langle \hat{\epsilon}^2_{\mathrm{X}M'\mathrm{i}} \rangle &= L_{px}^2 \langle \hat{\epsilon}^2_{\mathrm{X}M\mathrm{i}} \rangle + L_{pp}^2 \langle \hat{\epsilon}^2_{\mathrm{P}M\mathrm{i}} \rangle + L_{px}L_{pp} \langle (\hat{\epsilon}_{\mathrm{X}M\mathrm{i}}\hat{\epsilon}_{\mathrm{P}M\mathrm{i}} + \hat{\epsilon}_{\mathrm{P}M\mathrm{i}}\hat{\epsilon}_{\mathrm{X}M\mathrm{i}}) \rangle \end{aligned}$$

The fact that the M-measurement is retrodictively optimal and balanced means that $\langle \hat{\epsilon}^2_{\mathrm{X}M\mathrm{i}} \rangle = \langle \hat{\epsilon}^2_{\mathrm{P}M\mathrm{i}} \rangle = \frac{1}{2}$. Also, we have from Lemma 2, proved in ref. [5], that

$$(\hat{\epsilon}_{\mathrm{X}M\mathrm{i}} + i\hat{\epsilon}_{\mathrm{P}M\mathrm{i}})\,|\psi \otimes \phi_{\mathrm{ap}}\rangle = 0$$

for every initial system state $|\psi\rangle$ (where $|\phi_{\mathrm{ap}}\rangle$ is the initial apparatus state, as before). Hence

$$\left\langle (\hat{\epsilon}_{\mathrm{X}M\mathrm{i}}\hat{\epsilon}_{\mathrm{P}M\mathrm{i}} + \hat{\epsilon}_{\mathrm{P}M\mathrm{i}}\hat{\epsilon}_{\mathrm{X}M\mathrm{i}}) \right\rangle = -i \left\langle (\hat{\epsilon}_{\mathrm{X}M\mathrm{i}} + i\hat{\epsilon}_{\mathrm{P}M\mathrm{i}})^2 \right\rangle = 0$$

Consequently

$$\left\langle \hat{\epsilon}^2_{\mathrm{X}M'\mathrm{i}} \right\rangle = \frac{1}{2}\left(L_{xx}^2 + L_{xp}^2\right)$$
$$\left\langle \hat{\epsilon}^2_{\mathrm{X}M'\mathrm{i}} \right\rangle = \frac{1}{2}\left(L_{px}^2 + L_{pp}^2\right)$$

It follows that the M'-measurement is retrodictively optimal and balanced if and only if

$$L_{xx}^2 + L_{xp}^2 = L_{px}^2 + L_{pp}^2 = 1$$

which implies

$$LL^{\mathrm{T}} = \begin{pmatrix} 1 & L_{xx}L_{px} + L_{xp}L_{pp} \\ L_{xx}L_{px} + L_{xp}L_{pp} & 1 \end{pmatrix}$$

where L^{T} denotes the transpose of L. Since $\det L = 1$ we must have $L_{xx}L_{px} + L_{xp}L_{pp} = 0$, which means that L is a rotation matrix. We conclude, that the necessary and sufficient condition for the M'-measurement to be retrodictively optimal and balanced is that

$$M' = R_\psi M$$

for some ψ [where R_ψ is a rotation matrix as defined in Eq. (5)]. If M and M' satisfy this condition we will say that the corresponding measurements are informationally equivalent, and we will write $M \sim M'$.

If $M \sim M'$ it means that a retrodictively optimal measurement of $x_{\theta+\phi}$, $p_{\theta-\phi}$ to accuracies $\pm\frac{\lambda}{\sqrt{2\sec 2\phi}}$, $\pm\frac{1}{\lambda\sqrt{2\sec 2\phi}}$ yields exactly the same information as a retrodictively optimal measurement of $x_{\theta'+\phi'}$, $p_{\theta'-\phi'}$ to accuracies $\pm\frac{\lambda'}{\sqrt{2\sec 2\phi'}}$, $\pm\frac{1}{\lambda'\sqrt{2\sec 2\phi'}}$

The condition for $M \sim M'$ can alternatively be written

$$M'^{\mathrm{T}} M' = M^{\mathrm{T}} M \tag{11}$$

In other words, two measurements are informationally equivalent if and only if the corresponding matrices define the same phase-space metric [see Eq. (3)].

We conclude this section by showing, that given any $M \in \mathrm{SL}(2, \mathbb{R})$, it is possible to find $M_0 \sim M$ with zero obliquity (so that the axes are perpendicular).

We can write

$$M^{\mathrm{T}} M = \begin{pmatrix} a & c \\ c & b \end{pmatrix} \tag{12}$$

where a, $b > 0$ and $ab - c^2 = 1$. Let M_0 be a matrix with zero obliquity:

$$M_0 = \begin{pmatrix} \frac{1}{\lambda_0}\cos\theta_0 & \frac{1}{\lambda_0}\sin\theta_0 \\ -\lambda_0 \sin\theta_0 & \lambda_0\cos\theta_0 \end{pmatrix}$$

[see Eq. (2)]. We want to show that it is possible to choose λ_0, θ_0 so that

$$M_0^{\mathrm{T}} M_0 = M^{\mathrm{T}} M$$

which means

$$a - b = \left(\frac{1}{\lambda_0^2} - \lambda_0^2\right)\cos 2\theta_0$$
$$a + b = \left(\frac{1}{\lambda_0^2} + \lambda_0^2\right)$$
$$2c = \left(\frac{1}{\lambda_0^2} - \lambda_0^2\right)\sin 2\theta_0$$

It is readily confirmed that these equations are soluble. An explicit solution is

$$\theta_0 = \begin{cases} \frac{1}{2}\tan^{-1}\frac{2c}{b-a} & \text{if } b \neq a \\ -\frac{\pi}{4} & \text{if } b = a \end{cases} \tag{13}$$

and

$$\lambda_0 = \begin{cases} \frac{1}{\sqrt{2}}\left((a+b) - \operatorname{sign}(a-b)\left((a+b)^2-4\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} & \text{if } b \neq a \\ \sqrt{a+c} & \text{if } b = a \end{cases} \tag{14}$$

Using these formulae, it is straightforward to express θ_0, λ_0 directly in terms of the parameters θ, ϕ, λ. For the sake of simplicity we confine ourselves to the case $\lambda = 1$, when

$$\theta_0 = \begin{cases} [\theta]_{\frac{\pi}{2}} - \frac{\pi}{4} & \text{if } \phi \neq 0 \\ -\frac{\pi}{4} & \text{if } \phi = 0 \end{cases}$$

and

$$\lambda_0 = \begin{cases} \left(\sec 2\phi + \operatorname{sign}(\sin 2\theta)\tan 2\phi\right)^{\frac{1}{2}} & \text{if } \sin 2\theta \neq 0 \\ \left(\sec 2\phi + \cos 2\theta \tan 2\phi\right)^{\frac{1}{2}} & \text{if } \sin 2\theta = 0 \end{cases}$$

where the notation $[\theta]_{\frac{\pi}{2}}$ means "θ mod $\frac{\pi}{2}$":

$$[\theta]_{\frac{\pi}{2}} = \theta - \frac{n\pi}{2} \qquad \text{if } \frac{n\pi}{2} \leq \theta < \frac{(n+1)\pi}{2}$$

for every integer n.

This means that a retrodictively optimal measurement of the non-orthogonal quadratures $\hat{x}_{\theta+\phi}$, $\hat{p}_{\theta-\phi}$ to accuracies $\pm\frac{\lambda}{\sqrt{2\sec 2\phi}}$ and $\pm\frac{1}{\lambda\sqrt{2\sec 2\phi}}$ yields the same information as a measurement of the orthogonal quadratures $\hat{x}_{\theta_0}$, $\hat{p}_{\theta_0}$ to accuracies $\pm\frac{\lambda_0}{\sqrt{2}}$ and $\pm\frac{1}{\sqrt{2}\lambda_0}$. Fig. 2 gives an illustration, for the case $\lambda = 1$, $\theta = 0$, $\phi = 40^{\text{o}}$ [implying $\theta_0 = -45^{\text{o}}$ and $\lambda_0 = (\sec 80^{\text{o}} + \tan 80^{\text{o}})^{\frac{1}{2}}$].

4. The Distribution of Measured Values

Let M be any matrix $\in \mathrm{SL}(2,\mathbb{R})$, and consider a balanced retrodictively optimal measurement of the observables $\hat{x}_M$, $\hat{p}_M$. Let $\hat{\mu}_{\mathrm{X}M}$ and $\hat{\mu}_{\mathrm{P}M}$ be the pointer observables describing the outcome of this measurement. Define $\hat{\mu}_{\mathrm{X}}$, $\hat{\mu}_{\mathrm{P}}$ by

$$\begin{pmatrix} \hat{\mu}_{\mathrm{X}} \\ \hat{\mu}_{\mathrm{P}} \end{pmatrix} = M^{-1} \begin{pmatrix} \hat{\mu}_{\mathrm{X}M} \\ \hat{\mu}_{\mathrm{P}M} \end{pmatrix}$$

(*c.f.* Eq. (10)). Then $\hat{\mu}_{\mathrm{X}}$, $\hat{\mu}_{\mathrm{P}}$ are the pointer observables for a measurement of $\hat{x}$ and $\hat{p}$. This measurement will not be retrodictively optimal and balanced unless $M^{\mathrm{T}}M = 1$.

Let ρ_M be the probability density function describing the result of the measurement of $\hat{x}_M$, $\hat{p}_M$; and let Q_M be the probability density function describing the result of the measurement of $\hat{x}$, $\hat{p}$. Then

$$Q_M(x,p) = \rho_M(x_M, p_M)$$

for all x, p, where

$$\begin{pmatrix} x_M \\ p_M \end{pmatrix} = M \begin{pmatrix} x \\ p \end{pmatrix}$$

Let $\hat{a}_M$ and $|(x,p)_M\rangle$ be the annihilation operator and (normalised) squeezed state defined by

$$\hat{a}_M = \frac{1}{\sqrt{2}}\left(\hat{x}_M + i\hat{p}_M\right)$$
$$\hat{a}_M\big|(x,p)_M\big\rangle = \frac{1}{\sqrt{2}}\left(x_M + i\,p_M\right)\big|(x,p)_M\big\rangle \tag{15}$$

Using the result proved in ref. [5] we have

$$Q_M(x,p) = \rho_M(x_M, p_M) = \frac{1}{2\pi}\big\langle (x,p)_M\big|\,\hat{\rho}\,\big|(x,p)_M\big\rangle \tag{16}$$

where $\hat{\rho}$ is the density matrix describing the initial state of the system. We see from this that Q_M is a generalised Husimi function of the kind defined in Section 1.

It was shown in the last section that there exists ψ such that

$$M = R_\psi T_{\lambda_0} R_{\theta_0}$$

where λ_0, θ_0 are the quantities defined by Eqs. (13) and (14), and where R_ψ, T_{λ_0}, R_{θ_0} are the matrices defined by Eq. (5). In view of Eqs. (6-8) we then have [16]

$$\big|(x,p)_M\big\rangle = e^{i\chi}\hat{D}_{xp}\hat{U}^\dagger_{\theta_0}\hat{W}^\dagger_{\lambda_0}\hat{U}^\dagger_{\psi}\big|0\big\rangle = e^{i\chi}\hat{D}_{xp}\hat{U}^\dagger_{\theta_0}\hat{W}^\dagger_{\lambda_0}\big|0\big\rangle \tag{17}$$

where χ is a phase, $\hat{D}_{xp}$ is the displacement operator

$$\hat{D}_{xp} = \exp\left[-i(x_M\,\hat{p}_M - p_M\,\hat{x}_M)\right] = \exp\left[-i(x\hat{p} - p\hat{x})\right]$$

and where $|0\rangle$ is the vacuum state annihilated by $\hat{a} = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p})$. Hence

$$\begin{aligned} &Q_M(x,p) \\ &\quad = \frac{1}{\pi}\int dx'dp'\,\exp\left[-a(x'-x)^2 - 2c(x'-x)(p'-p) - b(p'-p)^2\right] W(x',p') \end{aligned} \tag{18}$$

where a, b, c are the quantities defined in Eq. (12) and W is the Wigner function describing the initial system state. It can be seen that the quadratic form occurring in the argument of the exponential is the phase space metric corresponding to M [*c.f.* Eqs. (3) and (12)]. This means that two retrodictively optimal measurements define the same distribution if and only if they are informationally equivalent—as was to be expected.

In terms of the parameters λ_0, θ_0 we have

$$Q_M(x,p) = \frac{1}{\pi}\int dx'dp'\,\exp\left[-\frac{1}{\lambda_0^2}(x'_{\theta_0} - x_{\theta_0})^2 - \lambda_0^2(p'_{\theta_0} - p_{\theta_0})^2\right] W(x',p')$$

It can be seen that Q_M is obtained by smoothing the Wigner function on the scale λ_0 parallel to the x_{θ_0} axis, and on the scale $\frac{1}{\lambda_0}$ parallel to the p_{θ_0} axis.

References

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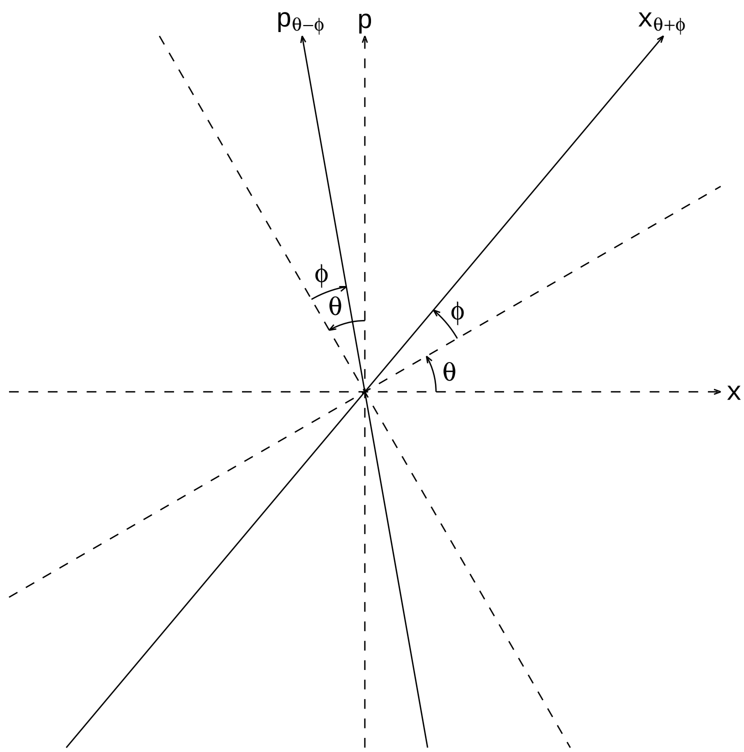

FIGURE 1. Oblique axes: parameterisation

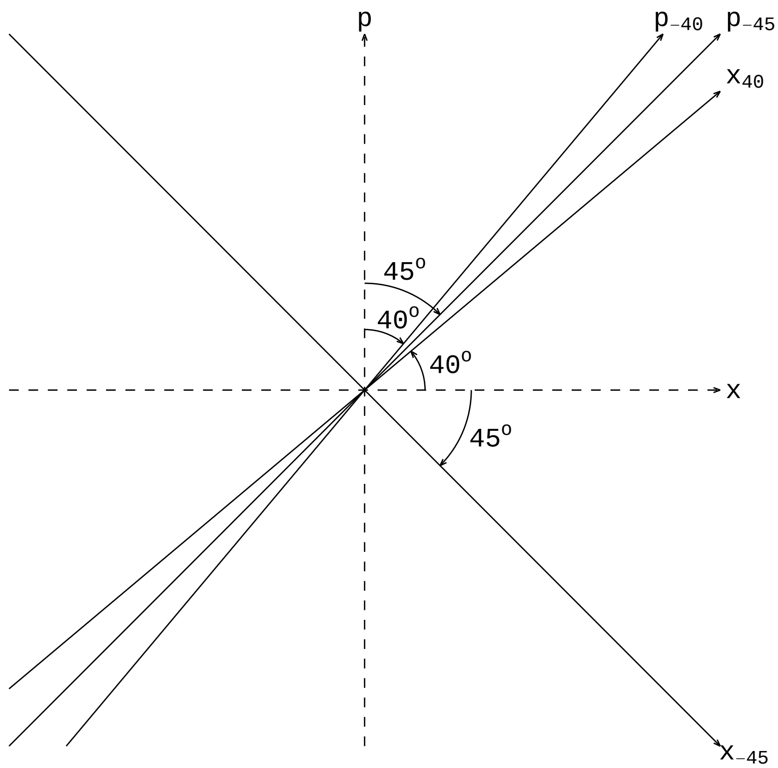

FIGURE 2. Equivalence of oblique and orthogonal phase space coordinate systems. Measuring the oblique coordinates x_{40} and p_{-40} each to the same accuracy $\pm\frac{1}{\sqrt{2\sec 80^o}} = \pm 0.29$ is equivalent to measuring the orthogonal coordinates x_{-45} and p_{-45} to accuracies $\pm\frac{\lambda_0}{\sqrt{2}} = \pm 2.39$ and $\pm\frac{1}{\sqrt{2}\lambda_0} = \pm 0.21$ respectively. The measurement of p_{-45} is more accurate than the measurements of x_{40} and p_{-40}. The measurement of x_{-45} is much less accurate.